

Minimum Entropy Production of Neutrino Radiation in the Steady State

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Received September 3, 1997; final May 7, 1998

A thermodynamic minimum principle valid for photon radiation is shown to hold for arbitrary geometries. It is successfully extended to neutrinos, in the zero mass and chemical potential case, following a parallel development of photon and neutrino statistics. This minimum principle stems more from that of Planck than that of classical Onsager–Prigogine irreversible thermodynamics. Its extension from bosons to fermions suggests that it may have a still wider validity.

KEY WORDS: Radiation; neutrinos; photons; fermions; bosons; minimum entropy production.

1. INTRODUCTION

Photon radiation has a distinctive quality in that it interacts little enough with matter that it is typically far from thermal equilibrium even when matter may be close to equilibrium. This is the origin of classical radiative transfer,^(1, 2) which represents a link between far-from-equilibrium properties of radiation to the near-equilibrium thermodynamics of matter, leading to some curious thermodynamical consequences.^(3, 4) These in particular concern minimum entropy production in a non-equilibrium steady state (NESS). This is a property distinct from the classical theory of irreversible thermodynamics,⁽⁵⁾ as few of the requirements for that theory hold: one does not even have a local thermodynamical bilinear form of entropy production.^(3, 6)

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Is this distinctive linkage one that may only be found in the domain of bosons? The answer is surely not, as neutrinos, which are fermions, are even more tenuously linked to matter than photons. However, a beam of neutrinos, unlike one of photons, is not at all the norm. In the case of fermions, in contrast, we must turn to the atypical domains of a supernova and the early Universe to have macroscopic beams interacting meaningfully at a thermodynamical level with matter. These physical situations involve a boundary in space or time between neutrino equilibrium and non-equilibrium. If we allow neutrino production without effective absorption, ordinary stars provide another example domain.

However, the question is to what extent the thermodynamical properties that hold for photons also hold for neutrinos. That is, is there a similar minimum principle for entropy production in the case of neutrino radiation? This paper addresses this question by generalizing “radiation” to include any exactly or nearly massless particles whose number is not conserved. We consider only the case of matter in local thermal equilibrium (LTE) and where macroscopic thermodynamics is valid, not the more general case when LTE breaks down and kinetic theory is necessary.⁽⁷⁾

We proceed here with the assumption that the rest mass and chemical potential of neutrinos are zero. Clearly a more comprehensive treatment must include the possibility that neither is zero.⁽⁸⁾ However there are advantages to proceeding in this manner in the first instance. By making this choice we ensure that the neutrinos are as much like the photons as possible. The rest mass is not so much a problem here as is the chemical potential. A non-zero chemical potential presents a qualitative departure from the thermodynamic properties of photons in that it implies an additional independent thermodynamical variable. However, what is learned in this case can be a guide to future work.

We find that neutrinos have an entropy production minimum principle in the steady state similar to that of photons, which also manifests itself as a conservation principle for energy. Implicit in weak reactions involving neutrinos is the conservation, not only of electric charge Q , but of lepton number L and baryon number B as well. Q , B , and L conservation in weak reactions play a non-trivial role, unlike Q in purely electromagnetic processes. These quantities are assumed to be exactly conserved in the microscopic sense. But conservation in this paper may also have a secondary thermodynamical meaning: absence of sources or sinks of these numbers in the form of macroscopic, thermodynamic reservoirs.

We proceed in a parallel manner between photons and neutrinos in order to highlight differences and similarities and to link with the previous work on photons.

II. GENERAL DEFINITIONS AND RELATIONS

Phase space for both neutrinos and photons is defined by position \mathbf{r} , momentum \mathbf{p} , implying energy ε . In the case of photons it is often customary to introduce frequency ν as a proxy for energy and wave number \mathbf{k} instead of momentum. The energy of unpolarized particles in the phase space volume $d^3p d^3r$ is

$$2n\varepsilon \frac{d^3p d^3r}{h^3} \quad (1)$$

where n is the mean occupation number for either photons or neutrinos. h is Planck's constant. The entropy of that same volume is

$$2k[\mp(1 \mp n) \ln(1 \mp n) - n \ln n] \frac{d^3p d^3r}{h^3} \quad (2)$$

where k is Boltzmann's constant. The upper signs correspond to neutrinos and the lower to photons.

A momentum-dependent temperature $T_{\mathbf{p}}$ may be introduced for this small phase volume, by forming the derivative of entropy with respect to energy,

$$\frac{1}{T_{\mathbf{p}}} = \frac{k}{\varepsilon} \ln(n^{-1} \mp 1) \quad (3)$$

which takes its physical meaning from the time-independent steady state of noninteracting quanta. This makes the cell indistinguishable from one that shares the same temperature with all other cells. We extend this naturally to neutrinos given the assumptions of the paper: the rest mass m_ν and the chemical potential μ_ν are both zero. One finds

$$\frac{d^3p d^3r}{h^3} = \left(\frac{kT_{\mathbf{p}}}{hc}\right)^3 x^2 dx d\Omega d^3r \quad (4)$$

given the new variable x such that $\varepsilon = pc$, $x = \varepsilon/kT_{\mathbf{p}}$, and $d^3p = p^2 dp d\Omega$, where $d\Omega$ is an element of solid angle.

In thermal equilibrium,

$$n_{\mathbf{p}} = \frac{1}{e^x \pm 1} \quad (5)$$

and we may drop the subscript \mathbf{p} on the temperature. The energy in the phase volume may then be written

$$\frac{2(kT)^4}{(hc)^3} \frac{x^3}{(e^x \pm 1)} dx d\Omega d^3r \quad (6)$$

From this we note that when an integration over x is carried out that the fourth-power law follows for both photons *and* neutrinos. The only difference between the two is in the numerical factor of the integral due to the “ \pm ” in the denominator in the integrand of the x integration. The entropy of the phase volume is treated similarly. After some simple manipulations it becomes

$$\frac{2k(kT)^3}{(hc)^3} \left[\frac{x^3}{(e^x \pm 1)} \pm x^2 \ln(1 \pm e^{-x}) \right] dx d\Omega d^3r \quad (7)$$

This implies the expected third-power behaviour for entropy in equilibrium, for neutrinos as well as photons.

The four integrals:

$$\int_0^\infty \frac{x^3}{(e^x \pm 1)} dx = \frac{15 \mp 1}{16} \left(\frac{\pi^4}{15} \right) \quad (8)$$

and

$$\int_0^\infty \pm x^2 \ln(1 \pm e^{-x}) dx = \frac{15 \mp 1}{16} \left(\frac{1}{3} \right) \left(\frac{\pi^4}{15} \right) \quad (9)$$

are easily deduced by simple series expansions. From these we find the energy per unit volume into solid angle $d\Omega$,

$$\frac{15 \mp 1}{16} \left(\frac{\sigma}{\pi c} \right) T^4 d\Omega \quad (10)$$

where $\sigma = 2k^4\pi^5/(15c^2h^3)$, the Stefan-Boltzmann constant. Similarly for the entropy,

$$\left(\frac{4}{3} \right) \frac{15 \mp 1}{16} \left(\frac{\sigma}{\pi c} \right) T^3 d\Omega \quad (11)$$

The vector flux density of energy into solid angle $d\Omega$ with direction $\hat{\mathbf{m}}$ is

$$\frac{15 \mp 1}{16} \left(\frac{\sigma}{\pi} \right) T^4 \hat{\mathbf{m}} d\Omega \quad (12)$$

and for entropy,

$$\left(\frac{4}{3} \right) \frac{15 \mp 1}{16} \left(\frac{\sigma}{\pi} \right) T^3 \hat{\mathbf{m}} d\Omega \quad (13)$$

The integrals (8, 9) give the canonical fermion factor of $7/8$ relative to bosons. We conclude that the flux density per solid angle, known variously as the specific intensity or radiance, is for energy,

$$\frac{15 \mp 1}{16} \left(\frac{\sigma}{\pi} \right) T^4 \quad (14)$$

and for entropy,

$$\left(\frac{4}{3} \right) \frac{15 \mp 1}{16} \left(\frac{\sigma}{\pi} \right) T^3 \quad (15)$$

Even out of equilibrium, we may relate the specific intensity I_ε , for a given ε , to n from Eqs. (1) and (4) and the *flux density* into a solid angle,

$$I_\varepsilon = \frac{2n_\varepsilon \varepsilon^3}{h^3 c^2} \quad (16)$$

The specific entropy intensity is

$$J_\varepsilon = \frac{2k\varepsilon^2}{h^3 c^2} \left[\mp \left(1 \mp \frac{c^2 h^3 I_\varepsilon}{2\varepsilon^3} \right) \ln \left(1 \mp \frac{c^2 h^3 I_\varepsilon}{2\varepsilon^3} \right) - \left(\frac{c^2 h^3 I_\varepsilon}{2\varepsilon^3} \right) \ln \left(\frac{c^2 h^3 I_\varepsilon}{2\varepsilon^3} \right) \right] \quad (17)$$

The fundamental extensive quantities are the specific entropy flux J_ε and specific energy flux I_ε . Note that expression (3) is recovered by forming $dJ_\varepsilon/dI_\varepsilon$.

Although neutrino number is not conserved, lepton number is, and it is thus physically important to define a specific *number* flux for neutrinos N_ε corresponding to I_ε :

$$N_\varepsilon = \frac{2n_{\mathbf{p}} \varepsilon^2}{h^3 c^2} \quad (18)$$

where

$$I_e = \varepsilon N_e \quad (19)$$

In the case of photons, number is not interesting, as photon number is not conserved. In the case of fermions, some number conservation law always holds (because of the half-integral spins). With zero chemical potential, however, the neutrino number is a purely auxiliary quantity and depends on the energy flux. Conversely, we could take neutrino number as fundamental and energy as derived; in either case, only one variable is independent. In the case of non-zero chemical potential, not treated in this paper, neutrino number and energy flux become independent variables, with an energy-dependent chemical potential μ_e .

III. ENTROPY PRODUCTION AND ENTROPY FLOWS

Entropy is an extensive thermodynamic property and can be localized and integrated to determine a global amount. Localization is also possible for entropy production itself. This result agrees with the general principle of the locality of physical interactions, so long as a thermodynamical picture is valid. It also means that thermodynamics is not restricted to a macroscopic box. If we divide space into distinct regions, boundary surfaces are defined; entropy can be moved between regions and the notion of entropy flux across a surface follows.

The entropy production rate can be expressed in a balance or conservation equation,

$$\varepsilon_s = \frac{\partial s}{\partial t} + \nabla \cdot \mathcal{F} \quad (20)$$

where ε_s is the entropy production rate per unit volume, s is the volume density of entropy, and \mathcal{F} is the entropy flux density.

It is only by convention that this equation is called a *conservation* equation. It is really an accounting equation, with no implication of conservation. In fact, this equation allows a general statement of the second law of thermodynamics: $\varepsilon_s \geq 0$.

A somewhat artificial distinction may be made now in Eq. (20), between radiative and matter processes. In the former case we necessarily must always consider full phase space, while in the latter we assume a near-equilibrium distribution in energy, nearly identical in all directions (so-called “local thermal equilibrium” or LTE). In that case we find

$$\varepsilon_s = \varepsilon_s^m + \varepsilon_s^r \quad (21)$$

where the superscript m denotes non-radiative components which shall be termed “matter.” γ denotes radiative components. Writing these entropy sources out explicitly we have,

$$\varepsilon_s = \frac{\partial s_m}{\partial t} + \frac{\partial s_\gamma}{\partial t} + \nabla \cdot \mathbf{Y}_s + \nabla \cdot \mathbf{H} \quad (22)$$

where s_m is the volume density of entropy in matter, and s_γ is the volume density of entropy in radiation. \mathbf{Y}_s and \mathbf{H} denote non-radiative and radiative entropy flux densities, respectively.

As radiation we mean here, of course, photons or neutrinos, while other particles take the role of being non-radiative and in LTE. A mixture of near-equilibrium (LTE) matter and the far-from-equilibrium radiation is a typical one in the Universe. It is the mixture in which you are immersed while reading this page: you are warm, and yet you can read these words radiatively.

By using the equations of state and balance equations for extensive variables we re-express the entropy production rate in the form⁽⁹⁾

$$\varepsilon_s = \sum_k \{a_k \varepsilon_k + \nabla a_k \cdot \mathbf{Y}'_k\} + \frac{\partial s_r}{\partial t} + \nabla \cdot \mathbf{H} \quad (23)$$

where the sum is over contributions from extensive variables with index k . a_k is the conjugate intensive variable divided by the temperature. ε_k is the creation rate of variable k (for example, the rate that internal energy is created from the radiation field, nuclear reactions, or viscous dissipation). \mathbf{Y}'_k is the flux density of variable k (for example, the flux density of internal energy in the case of diffusion). The prime denotes the value in the rest frame of the medium. As massless radiation is without a rest frame, the separation of the entropy production into these two parts thus turns out to be not at all artificial.

If we assume in (22) a steady radiation field, and integrate over a finite volume V bounded by a surface S , with element $d\mathbf{S}$, containing all of the matter, then the overall entropy production rate Σ is

$$\Sigma = \int_V \frac{\partial s_m}{\partial t} dV + \int_S \mathbf{H} \cdot d\mathbf{S} \quad (24)$$

because matter fluxes must vanish across S .

If we ignore matter transport processes, Eq. (23) becomes

$$\Sigma = \int_V \sum_k \{a_k \varepsilon_k\} dV + \int_S \mathbf{H} \cdot d\mathbf{S} \quad (25)$$

We may arrive at this result, alternatively, by imagining that the process is steady and that the entropy change of matter (the first integral) is reversibly drained off to a heat or another type of reservoir. That is, for this steady case, we deal only with a subsystem, thermodynamically speaking, and so conservation laws do *not* hold *macroscopically* (i.e., integrated) in the subsystem alone. Of course this has no bearing on *microscopic* conservation laws.

It is worth noting that the terms under the first integral of (25) are all due to processes in matter and are not a part of the entropy production for photons or neutrinos. It is a common misconception to interpret the radiation heating rate over the temperature, which is a possible term under the first integral, as the entropy production of radiation. It should be clear from this construction that $\int_V \varepsilon_s^\gamma dV$ is all accounted for through the second integral of (25).

IV. MINIMUM ENTROPY PRODUCTION

Equation (25) provides a structure for computing the entropy production rate due to the interaction of matter and radiation for many finite bodies locally in equilibrium. The first term represents whatever changes are manifest in the entropy of the body while the second term accounts for changes in the radiation field itself due to the interaction.

If photon radiation is impinging on a body of temperature T in a vacuum, the entropy production rate is from (25) just

$$\Sigma = \int_V \left(-\frac{\mathbf{\nabla} \cdot \mathbf{F}}{T} \right) dV + \int_S \mathbf{H} \cdot d\mathbf{S} \quad (26)$$

where the volume V is any containing the body, and \mathbf{F} is the flux density of energy radiation. The latter is the integral over all solid angles and energy of $I_e \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the unit vector defining the direction that I_e (Eq. (16)) flows in. If the temperature is (artificially held) uniform over the body, then

$$\Sigma = -\frac{1}{T} \int_S \mathbf{F} \cdot d\mathbf{S} + \int_S \mathbf{H} \cdot d\mathbf{S} \quad (27)$$

If the surface area of the body is A , and it emits as a black body, then

$$\Sigma = \frac{1}{T} \left\{ \left| \int_S \mathbf{F}^i \cdot d\mathbf{S} \right| - \sigma T^4 A \right\} - \left| \int_S \mathbf{H}^i \cdot d\mathbf{S} \right| + \frac{4}{3} \sigma T^3 A \quad (28)$$

where the remaining integrals represent impinging photon radiation, and the superscript i denotes an impinging flow only, which is independent of the state of the body.

It follows that

$$\frac{d\Sigma}{dT} = -\frac{1}{T^2} \left\{ \left| \int_S \mathbf{F}^i \cdot d\mathbf{S} \right| - \sigma T^4 A \right\} \quad (29)$$

or for a minimum

$$\left\{ \left| \int_S \mathbf{F}^i \cdot d\mathbf{S} \right| - \sigma T^4 A \right\} = 0 \quad (30)$$

That is, the entropy production rate is a minimum in the steady state, implying energy conservation,^(3, 10) but for an arbitrary geometry and impinging field.

Equations (26) to (30) could represent the entropy production of a (black) planet irradiated by photons originating elsewhere from a star, or could represent a gas cloud under similar photon radiation. They could also represent the entropy production of a target of matter irradiated by a laser. One can envisage matter in the laboratory as being connected to a heat bath, or circumstances where the heat capacity is very high, to justify the steady, isothermal construction of the rate of entropy change in matter (first term in (26)). Obviously the isothermal construction may be relaxed⁽¹⁰⁾ using the equations of Section III, but that is beyond the scope of this paper.

While these equations represent an extension of previous work^(3, 10) to arbitrary matter and radiation geometries, the remarkable thing about them is that they represent a minimum entropy production principle that is foreign to the classical minimum entropy production principle of Prigogine.⁽⁵⁾ There are no Onsager reciprocity relations, no "linear" empirical flux-force laws, and no meaningful thermodynamical fluxes and forces. In the latter regard, there need only be one temperature defined for the entire problem.

The question of whether this remarkable property is restricted to boson radiation in the form of photons has not been put previously, so we now consider the corresponding problem for fermions in the form of neutrinos. As in the case of photons we turn to Eq. (25), but at this point the differences between photons and neutrinos emerge, not in the second (radiation) term, but in the first (matter) term. That is because of the different manner in which neutrinos interact with matter. While photons do

not conserve their number and trivially conserve electric charge, neutrinos are linked to the conservation of charge, lepton number, and baryon number through the structure of weak interactions. Here we consider only first-generation fermions and only nucleons for hadrons:



and all related reactions. Thus for neutrinos (25) becomes

$$\Sigma = \int_V \left(-\frac{\nabla \cdot \mathbf{F}}{T} + \frac{\mu_e \dot{n}_e + \mu_n \dot{n}_n + \mu_p \dot{n}_p}{T} \right) dV + \int_S \mathbf{H} \cdot d\mathbf{S} \quad (32)$$

Recall that $\mu_\nu = 0$ is assumed for neutrinos. \dot{n}_e , \dot{n}_n , and \dot{n}_p represent rates of change of number densities for electrons, neutrons and protons respectively, each of which is multiplied by its corresponding chemical potential. Assuming chemical equilibrium within the matter,

$$\mu_e + \mu_p - \mu_n = 0 \quad (33)$$

This, together with an isothermal and black body (neutrino) assumption, leads to

$$\begin{aligned} \Sigma = & \frac{1}{T} \left\{ \left| \int_S \mathbf{F}^i \cdot d\mathbf{S} \right| - \frac{7}{8} \sigma T^4 A \right\} + \int_V \frac{1}{T} \{ \mu_n (\dot{n}_n + \dot{n}_p) + \mu_e (\dot{n}_e - \dot{n}_p) \} dV \\ & - \left| \int_S \mathbf{H}^i \cdot d\mathbf{S} \right| + \frac{7}{8} \cdot \frac{4}{3} \sigma T^3 A \end{aligned} \quad (34)$$

Conservation of baryon number and charge produce

$$\Sigma = \frac{1}{T} \left\{ \left| \int_S \mathbf{F}^i \cdot d\mathbf{S} \right| - \frac{7}{8} \sigma T^4 A \right\} - \left| \int_S \mathbf{H}^i \cdot d\mathbf{S} \right| + \frac{7}{8} \cdot \frac{4}{3} \sigma T^3 A \quad (35)$$

Except for the factors of (7/8), this equation is identical to (28). Thus in minimum entropy production, the energy balance steady state

$$\left\{ \left| \int_S \mathbf{F}^i \cdot d\mathbf{S} \right| - \frac{7}{8} \sigma T^4 A \right\} = 0 \quad (36)$$

must hold as well. Thus we find that this distinctive minimum entropy production result is extended to neutrinos, and thereby beyond the limitation to bosons to at least some fermions.

V. LOCAL AND NONLOCAL REGIMES

Generally, the interactions of neutrinos and photons with matter are most simply viewed as purely local. For statistical physics, however, we also need to count momentum states. If we make the matter-radiation separation of Section 3 and further assume LTE for matter, the matter momentum states can be integrated out, leaving the full phase space only for radiation. This condition allows us to avoid a full kinetic theory calculation.⁽⁷⁾ At this juncture, we have a choice of local versus action-at-a-distance representation for the radiation.

While the radiation exists in its own right, in the event that the overall entropy of the radiation field is not changing, we need only be concerned with matter-matter interactions mediated by radiation, as the radiation terms can be then be integrated out. Radiation then becomes merely a special kind of nonlocal heat transport, and Σ may be represented in a multi-local form. This multilocal form is at least true for the first term in (25) even when separate changes do take place in the radiation field, such as in conservative scattering processes.⁽⁶⁾ In all cases, it is the *total* Σ that is minimised.

In the case of quanta in an extended, continuous medium, a common matter-radiation LTE is often valid, with a common matter-radiation temperature $T(\mathbf{r})$. This temperature in general is not constant in space. LTE holds to extremely high accuracy for *photons* inside the photospheric surface of a star, for example, but not for neutrinos. The entropy production associated with the production and transport (diffusion) of photons is

$$\Sigma_\gamma = \int dV \left\{ (1/2) [4acT^5 / (3\kappa_\gamma)] [\nabla(1/T)]^2 + \varepsilon_\gamma / T \right\} \quad (37)$$

where ε_γ is the photon energy production rate density and κ_γ is the opacity (inverse mean free path) of the matter against photon diffusion.⁽¹¹⁾ Equation (37) corresponds to the second term in Eq. (25), but written as a volume integral of a divergence, up to but *not* including the photosphere. This is in contrast to photon entropy production discussed previously in this paper, in that the photons here are virtually in equilibrium with matter and so are diffusive, not radiative.

At the photospheric surface, a single LTE ceases to hold (see below), and the diffusive approximation of Eq. (37) breaks down. Nonetheless the second term of (25) still represents the entropy production in the radiation field, but at the photosphere and outside, the photons become radiative.

The complete photon entropy production of a star of radius R (including its photosphere) is

$$\Sigma_{\gamma, \text{sur}} = \frac{4}{3} \sigma T_{\text{sur}}^3 (4\pi R^2) \quad (38)$$

for the photon radiation released into empty space at an idealised sharp surface. (T_{sur} is the photospheric surface temperature.) As the volume of integration is increased Σ_{γ} picks up additional contributions to interactions with more matter, for example, with a planet.^(3, 4, 10, 12)

Neutrinos emitted by ordinary stars are quite different from photons: as the interior temperatures are not high enough for weak interactions to be in LTE, the neutrinos are not emitted in anything like a blackbody distribution, and are not subsequently thermalised. Their spectra are instead determined almost exactly by the microscopic reaction spectra and emerge essentially unaffected by the neutrinos' subsequent travel through stellar matter to empty space. If the emitting star does not absorb neutrinos and the receiving Earth-bound detector does not emit neutrinos, the total neutrino entropy production is

$$\Sigma_{\nu} = \int_{\text{emitter}} dV \int d^3p \frac{\varepsilon_{\mathbf{p}} \dot{n}_{\mathbf{p}}}{T_{\mathbf{p}}} - \int_{\text{receiver}} dV' \int d^3p' \frac{\varepsilon_{\mathbf{p}'} \dot{n}_{\mathbf{p}'}}{T_{\mathbf{p}'}} \quad (39)$$

where $\dot{n}_{\mathbf{p}}$ ($\dot{n}_{\mathbf{p}'}$) is the neutrino production (absorption) rate density in real and momentum space. In a NESS, $\dot{n}_{\mathbf{p}}$ depends only on $\varepsilon_{\mathbf{p}}$, not on emission direction $\hat{\mathbf{p}}$.

In an equilibrated supernova or the early Universe, on the other hand, the neutrinos are emitted and absorbed in LTE. The entropy production below the supernova neutrinosphere is a function of a single local temperature:

$$\Sigma_{\nu} = \int dV \{ (1/2) [7acT^5 / (6\kappa_{\nu})] [\nabla(1/T)]^2 + \varepsilon_{\nu}/T \} \quad (40)$$

like (37), with a neutrino mean free path $1/\kappa_{\nu}$ and an extra factor of $7/8$ in the diffusion part. The total Σ_{ν} including the neutrinosphere is analogous to (38). An analogous expression can be constructed for photon and neutrino entropy production in the early Universe before their respective decouplings, but this would require inclusion of general relativity.

Even if the neutrinos or photons are emitted and absorbed locally as a gas, the system in general is not in equilibrium with a single temperature T or $T(\mathbf{r})$. Multiple temperatures can be defined if each system component retains its own LTE, with each component spectrum thermal. For example,

a photon gas with a frequency-dependent temperature T_γ , may interact with matter of temperature T . Then

$$\Sigma_\gamma = \int dV \int d\varepsilon \int d\Omega I_\varepsilon(\mathbf{r}, \varepsilon, \Omega) \left[\frac{1}{T_\gamma(\mathbf{r}, \varepsilon, \Omega)} - \frac{1}{T(\mathbf{r})} \right] \quad (41)$$

I_ε is the local specific energy intensity of photons emitted by the matter. If $T_\gamma > T$, then $I_\varepsilon \leq 0$; if $T_\gamma < T$, then $I_\varepsilon \geq 0$. Thus Σ_γ is always ≥ 0 . Stellar atmospheres provide a related example. The radiation has a temperature $T_\gamma(\mathbf{r})$, while the various chemical species X_l each can have their own $T_l(\mathbf{r})$. Thus

$$\Sigma_\gamma = \sum_l \int dV \int d\varepsilon \int d\Omega I_\varepsilon(\mathbf{r}, \varepsilon, \Omega) \left[\frac{1}{T_\gamma(\mathbf{r}, \varepsilon, \Omega)} - \frac{1}{T_l(\mathbf{r})} \right] \quad (42)$$

Again I_ε is the local specific radiation intensity emitted by the matter. Contributions such as (41) or (42) occur in addition to such gradient terms as (37) or (40).

In a supernova,⁽¹³⁾ the matter and the neutrinos can have different temperatures T and T_ν , and Σ_ν picks up a contribution analogous to (41). The total Σ again has other contributions, e.g., from gradients of T , T_γ , and T_ν such as (37) and (40).

Neutrinos can interact among themselves by weak neutral currents and change their own phase space distribution without any ordinary matter present. The associated entropy production is

$$\begin{aligned} \Sigma_\nu^{\text{NC}} = & \int dV \int d\varepsilon \int d\Omega \int d\varepsilon' \int d\Omega' I_{\varepsilon\varepsilon'}(\mathbf{r}, \varepsilon, \varepsilon', \Omega, \Omega') \\ & \times \left[\frac{1}{T_\nu(\mathbf{r}, \varepsilon, \Omega)} - \frac{1}{T_\nu(\mathbf{r}, \varepsilon', \Omega')} \right] \end{aligned} \quad (43)$$

which is local in form. $I_{\varepsilon\varepsilon'}(\mathbf{r}, \varepsilon, \varepsilon', \Omega, \Omega')$ is the local doubly-specific radiation intensity of the neutrinos “shining” on themselves and is proportional to the neutrino-neutrino weak neutral current reaction cross section. Σ_ν^{NC} vanishes if thermal equilibrium obtains and there is only a single temperature, $T_{\mathbf{p}} = T_{\mathbf{p}'}$ for all \mathbf{p}, \mathbf{p}' , at each point \mathbf{r} .

VI. SUMMARY AND CONCLUSION

A non-zero density of entropy production, Σ , indicates a local process of an irreversible nature. It is distinct from the local density of entropy, and

it has no direct implications for processes elsewhere. For any given process, one may measure local nearness to equilibrium by its magnitude, although relative nearness of two different processes may have no meaning by this measure. If a single process is free, one expects that the entropy production rate will relax toward zero. If some exterior conditions prevent this dynamical relaxation, the process will instead simply relax as far as the exterior arrangements permit.

This dynamical argument anticipates minimum entropy production, at least for sufficiently simple systems where thermodynamic quantities have meaning. But it also anticipates, as thermodynamics often does, an analogous static construction of the same principle, as the relaxation may be imagined to be constrained to quasi-steady conditions in whatever state it may be.

Thus the principle of minimum entropy production found here is not unexpected. However in many respects it is very different from the classical minimum entropy production principle of Prigogine.⁽⁵⁾ It does not depend on Onsager's principle; there are no "linear" empirical flux-force laws, nor is there even a meaningful bilinear construction.

Under the LTE-NESS assumption, systems of both matter⁽⁵⁾ and photons^(3, 4, 10, 11) exhibit some type of minimum entropy production. A photon-like principle holds, under the same assumptions, for neutrinos, as seen in examples given in Sections IV and V. These examples can be generalized to many local or continuously varying temperatures.

Conservation laws play identical roles in all three cases, by constraining the microscopic interactions of the quanta. Thus, microscopic energy and momentum are always conserved, with all the appropriate macroscopic consequences. Because neutrinos and photons are both taken here as massless and not conserved in number, the analogy between these two particles can be carried through in most aspects. However, neutrinos are fermions, which always have some associated conservation law; in this case, lepton number L . Because the weak interactions conserve $B - L$, baryon number B is also conserved. Electromagnetic interactions conserve charge, but since photons themselves do not carry charge, this conservation law is trivial in radiative transfer, in contrast to the situation for neutrinos, which do carry L .

The exact masslessness of neutrinos is not proven experimentally,⁽¹⁴⁾ and a logical generalization of our results here is to extend the treatment to massive neutrinos. Although we have used an electron chemical potential μ_e in matter, another generalization left open is to include a neutrino chemical potential μ_ν . These two extensions will be presented in a subsequent publication.⁽⁸⁾

ACKNOWLEDGMENTS

We thank the Telluride Summer Research Center, where part of this work was done. D. C. Kennedy acknowledges the support of the University of Florida/Institute for Fundamental Theory, the U.S. Department of Energy under contract DE-FG05-86-ER40272 (U. Florida) and the NASA/Fermilab Theoretical Astrophysics group under DOE/NASA contract NAG5-2788.

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Communicated by J. L. Lebowitz